Journal of Sound and Vibration (1997) **203**(1), 127–150

THE EFFECTS OF SAMPLING RATE AND LENGTH OF AN ADAPTIVE FILTER ON THE ACTIVE CONTROL OF A PLANE SOUND WAVE IN A LOSSY SEMI-INFINITE WAVEGUIDE

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(*Received* 11 *July* 1996, *and in final form* 8 *November* 1996)

The active control of a plane sound wave in a semi-infinite waveguide can require a controller with a long weighting function. In a real system constrained to act on-line, the signals are sampled and the weighting function is truncated. Both the sampling rate and the number of coefficients of the adaptive digital filter acting as controller have important effects on active attenuation of noise. The optimal truncated weighting function is therefore determined in order to compare their respective effects. The optimal set-up point of these parameters is then found for a primary excitation consisting of a white noise convolved with an ideal low pass filter. It is shown that the optimal active attenuation depends on two non-dimensional parameters that are composed of four quantities: the speed of the processor, the loss coefficient in the waveguide, the location of the secondary source and the cut-off frequency of the low pass filter.

1. INTRODUCTION

The control of a plane wave in an acoustic waveguide has been identified by Lueg [1] in 1934 as the first candidate application for active noise control. It has however only been during the eighties that the development of adaptive signal processing has enabled applications to be made in an industrial duct system [2].

One of the most common adaptive algorithm for the control of a stationary random excitation is the filtered X-LMS presented by Widrow *et al*. [3, 4]. A detection sensor provides a reference signal which is supposed to be well correlated with the primary noise whereas an error sensor provides the error signal which has to be cancelled. Information coming from these two signals enables the adaptive filter of the controller to be updated. The convolution of the reference signal with this adaptive filter gives finally the electric signal that feeds into the secondary source.

Several physical and signal processing effects are known to limit the efficiency of active noise control [5]. First, systems are constrained to act *causally* with respect to the primary source: i.e., the secondary source cannot emit a signal before the detection sensor has detected a primary excitation. So the weighting function must be zero for negative times. The efficiency of an active control has been assessed by Nelson *et al*. [6] as a function of the acoustic delay and "predictability" of the primary source output. Moreover electrical delays due to anti-aliasing and the reconstruction filter must be added to the acoustic propagation delay. So the different electronical delays must be less than the physical propagation time between the detection sensor and the secondary source to ensure the

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constraint of causality. These practical components impose therefore a geometric constraint on the active noise control system [7].

Second, systems are constrained to act *on-line*. That means that digital filtering and adaptation must usually be performed by the processor in a computation time less than the sample time. This constraint imposes a limitation on the number of coefficients of the adaptive filter involved in the filtered X-LMS. So the weighting function is truncated: i.e., it is zero for times greater than its length. The sampling imposes furthermore the use of anti-aliasing filters.

The sampling rate and the length of the adaptive filter both influence the results of active control. Duhamel [8] evaluated their effects in the case of a control around a noise barrier. The present work is devoted to the detailed study of their respective influences on the active control of a plane sound wave in a semi-infinite lossy waveguide with a stationary random excitation. The choice of the reference signal in a semi-infinite waveguide conditions the shape of the weighting function. In the case of an independent reference signal (i.e., receiving no feedback coming from the secondary source) the control requires a long weighting function. The truncation of the weighting function can therefore largely reduce the active attenuation.

The objective of this work is to determine the optimal truncated weighting function of the controller in order to predict the active attenuation with respect to frequency. It is pointed out that the truncation limits this attenuation in narrow bands of frequencies. The use of anti-aliasing filters then forbids any control at frequencies greater than half the sampling frequency.

When the primary excitation is a white noise convolved with an ideal low pass filter, some requirements are presented for the set-up of the parameters (sampling rate, length of the adaptive filter). It is shown that the optimal active attenuation depends on two non-dimensional parameters that are composed of four quantities: the speed of the processor, the loss coefficient in the waveguide, the location of the secondary source and the cut-off frequency of the low pass filter.

2. DEFINITION OF THE REFERENCE AND ERROR SIGNALS

Let t denote the time and ω the angular frequency. Consider a semi-infinite lossy waveguide with a square cross-section of area S equal to a^2. A point monopolar primary source with volume velocity $q_p(t)$ is located at the upstream termination of the waveguide. A point monopolar secondary source with volume velocity $q_s(t)$ is located at the distance x_0 from the upstream termination (see Figure 1). The upstream termination is supposed

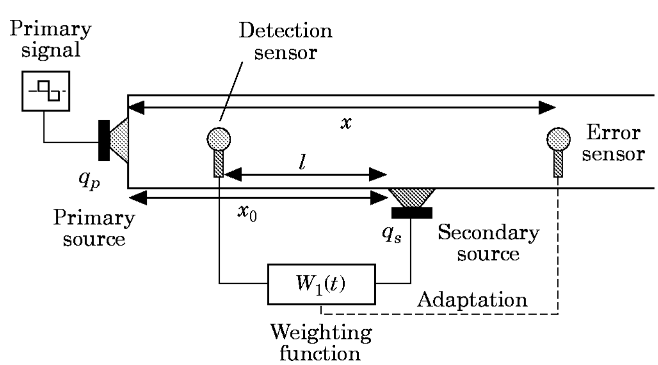

Figure 1. Active noise control system with feedback.

to be a perfectly reflective termination whereas the downstream termination is anechoic. Let $\tilde{k}$ denote the wavenumber, equal to $\omega/C_0 + \mathrm{j}\alpha$ where C_0 is the speed of sound, j the imaginary unit and α the loss coefficient, which is positive ($\alpha > 0$) and assumed independent of frequency to simplify the problem. The real part of $\tilde{k}$ is denoted by k. Let x be the distance from the upstream termination and $p(x, t)$ the sound pressure field. The dimension a in the cross-section is assumed small with respect to the wavelength λ of the signal so that the sound pressure can be considered independent of the other co-ordinates y and z. The Fourier transform of the sound pressure $p(x, t)$ is

$$p(x, \omega) = \int_{-\infty}^{\infty} p(x, t)\, \mathrm{e}^{\mathrm{j}\omega t}\, \mathrm{d}t. \tag{1}$$

The Fourier transforms of the volume velocities $q_p(t)$ and $q_s(t)$ are $q_p(\omega)$ and $q_s(\omega)$. The sound pressure field $p(x, \omega)$ satisfies the Helmholtz equation and two boundary value equations:

$$\frac{\mathrm{d}^2 p}{\mathrm{d}^2 x}(x, \omega) + \tilde{k}^2 p(x, \omega) = \rho_0 \mathrm{j}\omega \frac{q_s(\omega)}{S}\, \delta_{x_0},$$

$$(\mathrm{d}p/\mathrm{d}x)(0, \omega) = \rho_0 \mathrm{j}\omega q_p(\omega)/S, \qquad (\mathrm{d}p/\mathrm{d}x)(\infty, \omega) = \mathrm{j}\tilde{k} p(\infty, \omega). \tag{2}$$

δ_{x_0} is the unidimensional Dirac delta function at point x_0. The solution of the problem (2) has the following expression (see Appendix A for the details of the solution):

$$p(x, \omega) = [q_p(\omega)/S] Z_p(x, \omega) + [q_s(\omega)/S] Z_s(x, \omega),$$

$$Z_p(x, \omega) = \frac{\rho_0 C_0}{1 + \mathrm{j}(C_0 \alpha/\omega)}\, \mathrm{e}^{\mathrm{j}\tilde{k}x} \quad \forall x, \qquad Z_s(x, \omega) = Z_p(x, \omega) \cos(\tilde{k} x_0) \quad \forall x \geqslant x_0,$$

$$Z_s(x, \omega) = Z_p(x_0, \omega) \cos(\tilde{k} x) \quad \forall x \leqslant x_0. \tag{3}$$

If the sensors are unidirectional an index $^+$ is used to indicate that the sensors detect only downstream waves:

$$p^+(x, \omega) = [q_p(\omega)/S] Z_p(x, \omega) + [q_s(\omega)/S] Z_s^+(x, \omega),$$

$$Z_s^+(x, \omega) = Z_p(x, \omega) \cos(\tilde{k} x_0) \quad \forall x \geqslant x_0, \qquad Z_s^+(x, \omega) = Z_p(x_0, \omega)\, \mathrm{e}^{\mathrm{j}\tilde{k}x}/2 \quad \forall x \leqslant x_0. \tag{4}$$

2.1. REFERENCE SIGNAL WITH FEEDBACK

Let an error signal be denoted as $E(\omega)$ and a reference signal as $X(\omega)$. The error sensor is supposed to be an ideal omnidirectional sensor located at a distance x (with $x \geqslant x_0$) from the upstream termination. The detection sensor is also supposed to be an ideal omnidirectional sensor located at a distance l upstream from the secondary source. With the previous notations, the signals $E(\omega)$ and $X(\omega)$ have the following expressions:

$$E(\omega) = p(x, \omega), \qquad X(\omega) = p(x_0 - l, \omega). \tag{5}$$

Let $W_1(\omega)$ denote the transfer function of the controller between the reference signal $X(\omega)$ and the volume velocity $q_s(\omega)$ of the secondary source.

$$W_1(\omega) = \frac{q_s(\omega)}{X(\omega)} = \frac{S q_s(\omega)}{q_p(\omega) Z_p(x_0 - l) + q_s(\omega) Z_s(x_0 - l)}. \tag{6}$$

One is interested in the expression of this transfer function $W_1(\omega)$ when the error signal $E(\omega)$ is cancelled. If $E(\omega)$ is equal to zero, $q_p(\omega) = -q_s(\omega)\cos(\tilde{k}x_0)$. This leads to

$$W_1(\omega) = S/[Z_s(x_0 - l) - \cos(\tilde{k}x_0)Z_p(x_0 - l)]$$

$$= \frac{S}{[\cos(\tilde{k}(x_0 - l))Z_p(x_0) - \cos(\tilde{k}x_0)Z_p(x_0 - l)]}$$

$$= \frac{1 + \mathrm{j}C_0\alpha/\omega}{\rho_0 C_0}\frac{S}{\mathrm{j}\sin(\tilde{k}l)}. \tag{7}$$

This transfer function is close to the result of Eghtesadi and Levanthall [9]. This result is actually independent of the type of terminations and is also valid for an infinite waveguide. As pointed out by Nelson and Elliott [5], the controller has a large transfer function at certain frequencies and a long weighting function. The transfer function when the loss coefficient is small (i.e., $1/\alpha$ is large with respect to the wavelength λ) is

$$W_1(\omega) = -\frac{S}{-\rho_0 C_0}\frac{2}{\mathrm{e}^{-\mathrm{j}\tilde{k}l} - \mathrm{e}^{\mathrm{j}\tilde{k}l}} = -\frac{S}{\rho_0 C_0}\frac{2\mathrm{e}^{\mathrm{j}\tilde{k}l}}{1 - \mathrm{e}^{2\mathrm{j}\tilde{k}l}}. \tag{8}$$

The binomial theorem enables one to expand the expression $(1 - z)^{-1}$ as a series of the form $\Sigma_{i=0}^{\infty} z^m$ if $|z| < 1$. With z equal to $\mathrm{e}^{\mathrm{j}\tilde{k}l}$, one has $|z| < 1$ since $\alpha > 0$. The expansion of the denominator of equation (8) then gives

$$W_1(\omega) = -\frac{2S}{\rho_0 C_0}\sum_{i=0}^{\infty} \mathrm{e}^{(2i+1)\mathrm{j}\tilde{k}l}. \tag{9}$$

The inverse Fourier transform of the transfer function of the controller gives the following weighting function:

$$W_1(t) = -\frac{2S}{\rho_0 C_0}\sum_{i=0}^{\infty} \mathrm{e}^{-(2i+1)\alpha l}\,\delta(t - (2i+1)t_l) \tag{10}$$

where δ is the Dirac delta function and $t_l = l/C_0$. The weighting function is written as an infinite series of Dirac delta functions whose amplitude exponentially decreases with time.

The two drawbacks of the previous system (large transfer function at certain frequencies and a long weighting function) can be removed by the use of an ideal unidirectional detection sensor instead of an omnidirectional detection sensor. One can then rewrite the expression for the error signal, the detection signal and the transfer function W_1^+ of the controller with respect to the angular frequency as

$$E(\omega) = p(x, \omega), \qquad X(\omega) = p^+(x_0 - l, \omega), \tag{11}$$

$$W_1^+(\omega) = q_s(\omega)/X(\omega) = Sq_s(\omega)/[q_p(\omega)Z_p(x_0 - l) + q_s(\omega)Z_s^+(x_0 - l)], \tag{12}$$

and one is now interested in the expression for this new transfer function $W_1^+(\omega)$ when the error signal $E(\omega)$ is cancelled.

$J(W_g^2)$, representing the expected value of the squared sound pressure with the transfer function $W_2^g(\omega)$ of the controller:

$$J(W_2^g) = \mathrm{E}[p^2(x, t)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_p(x, \omega)\,\mathrm{d}\omega$$

$$= A_p\left(\frac{\rho_0 C_0}{S}\right)^2 \frac{\mathrm{e}^{-2\alpha x}}{2\pi}\int_{-\omega_f}^{\omega_f} [1 + W_2^g(\omega)\cos(\tilde{k}x_0)|^2\,\mathrm{d}\omega. \tag{65}$$

By introducing the expression for the transfer function $W_2^g(\omega)$ in equation (65), $J(W_2^g)$ can be determined (see Appendix C) as

$$J(W_2^g) = A_p\left(\frac{\rho_0 C_0}{S}\right)^2 \mathrm{e}^{-2\alpha x}\left\{\frac{2}{t_f} - \frac{2}{t_{\max}}\frac{\mathrm{sh}\,[2(M_w + 1)\alpha x_0]}{\mathrm{sh}\,2(M_w + 2)\alpha x_0]}\,\mathrm{e}^{2\alpha x_0}\right\}. \tag{66}$$

Here $t_{\max} = \max(t_f, 2t_g)$ and it has been supposed that the ratio $2t_0/t_{\max}$ is an integer, in order to simplify the expression for the objective function. If this ratio is not an integer but large enough, this expression gives a good estimate of $J(W_2^g)$.

The active attenuation γ can now be found. This is the ratio of the value of the objective function without control to its value with control. Its expression in decibels is

$$\gamma = 10\log_{10}[J(0)/J(W_2^g)] = -10\log_{10}\left[1 - \frac{t_f}{t_{\max}}\frac{\mathrm{sh}\,[2(M_w + 1)\alpha x_0]}{\mathrm{sh}\,[2(M_w + 2)\alpha x_0]}\,\mathrm{e}^{2\alpha x_0}\right]. \tag{67}$$

One can now introduce four non-dimensional numbers:

$$T_w = t_w/t_0 \text{ (time length)}, \qquad T_g = t_g/t_f \text{ (sampling period)};$$

$$\alpha_0 = \alpha x_0 \quad \text{(loss coefficient)}, \qquad V_{DSP} = t_f^2 f_{DSP}/t_0 \text{ (speed of the processor)}. \tag{68}$$

The active attenuation γ depends on three of them (T_g, T_w and α_0):

$$\gamma = -10\log_{10}[1 - G_g(T_g)G_w(T_w, \alpha_0)]. \tag{69}$$

$$G_g(u) = \frac{1}{\max(1, 2u)}, \qquad G_w(u, \alpha_0) = \mathrm{e}^{2\alpha_0}\frac{\mathrm{sh}\,\{2[M_w(u) + 1]\alpha_0\}}{\mathrm{sh}\,\{2[M_w(u) + 2]\alpha_0\}}, \qquad M_w(u) = E\left[\frac{u}{2} - \frac{1}{2}\right]. \tag{70}$$

Equation (69) shows clearly that the active attenuation γ depends separately on two factors: the non-dimensional sampling period T_g and the non-dimensional time length T_w of the weighting function. Their respective effects are summed up in the variations of the functions G_g and G_w. These variations are presented in Figure 6. On the one hand the function G_g decreases from 1 to 0 when the non-dimensional sampling period increases. On the other hand the function G_w increases from 0 to 1 when the non-dimensional time length of the controller increases. Whereas G_g is independent of the non-dimensional loss coefficient α_0, G_w is all the larger since this loss coefficient is large. It is known indeed that a large loss coefficient reduces the detrimental effect of the upstream reflection. At first sight, the optimal set-up point consists of a small non-dimensional sampling period T_g and a long non-dimensional time length T_w. Unfortunately, when the active system is constrained to act *on-line*, equation (63) must be satisfied and it forbids this ideal set-up point.

focus on the problem with an independent reference signal and find in this case the transfer function $W_2(\omega)$ of the controller:

$$W_2(\omega) = q_s(\omega)/q_p(\omega) = -1/\cos(\tilde{k}x_0)$$
$$= -2/(e^{-j\tilde{k}x_0} + e^{j\tilde{k}x_0}) = -2e^{j\tilde{k}x_0}/(1 + e^{2j\tilde{k}x_0}). \tag{17}$$

Again using the binomial theorem in order to expand the denominator of equation (17) yields

$$W_2(\omega) = 2\sum_{i=0}^{\infty}(-1)^i\, e^{(2i+1)j\tilde{k}x_0}. \tag{18}$$

The Fourier transform of the transfer function of the controller gives the weighting function

$$W_2(t) = -2\sum_{i=0}^{\infty}(-1)^i\, e^{-(2i+1)\alpha x_0}\,\delta(t-(2i+1)t_0), \tag{19}$$

where t_0 is equal to x_0/C_0.

Like $W_1(\omega)$, the transfer function $W_2(\omega)$ is large at certain frequencies. These frequencies are here equal to $f_m = (2m+1)/4t_0$ where m is an integer. Stell and Bernhard [12] showed that the active control is limited at these frequencies because the evanescent modes are largely excited. These considerations were verified experimentally by Laugesen and Johannesen [13].

Like $W_1(t)$, the weighting function $W_2(t)$ is long. In an *on-line* system, this weighting function must be truncated. In the next section, it is shown that this truncation limits the efficiency of the active control in narrow bands of frequencies. The calculations are developed only for the independent reference signal (weighting function $W_2(t)$) but could be adapted for the case with feedback (weighting function $W_1(t)$).

3. TRUNCATION OF THE WEIGHTING FUNCTION

If an active control system is constrained to act *causally* and *on-line*, there is not complete freedom of choice of the weighting function $W(t)$. The first constraint, coming from the causality, imposes a zero function for negative times:

$$W(t) = 0, \qquad t < 0. \tag{20}$$

Once the sampling rate is fixed, the digital filtering and adaptation must be performed in a computation time less than the sampling period. If this constraint is satisfied, the system is said to be *on-line*. The computation time depends directly on the number of coefficients of the FIR filters that represent, in the filtered X-LMS, on the one hand the weighting function of the controller and on the other hand the secondary path. In short an *on-line* active system imposes a maximum number of coefficients for the digital filters and a finite time length for the impulse responses. With t_w denoting the finite time length of the weighting function $W(t)$, one has

$$W(t) = 0, \qquad t > t_w. \tag{21}$$

3.1. APPROXIMATE WEIGHTING FUNCTION OF THE CONTROLLER

Consider first an approximate solution $W_2^o(t)$ that is the result of the truncation of the solution $W_2(t)$ with infinite time response:

$$W_2^o(t) = W_2(t) \quad \text{for } 0 \leqslant t \leqslant t_w, \quad W_2^o(t) = 0 \quad \text{for } t < 0 \text{ or } t > t_w. \tag{22}$$

The expression for $W_2^o(t)$ is found easily to be

$$W_2^o(t) = -2 \sum_{l=0}^{M_w} \mathrm{e}^{-(2i+1)\alpha x_0}(-1)^i\,\delta(t-(2i+1)t_0), \tag{23}$$

where $M_w = E[\frac{1}{2}(t_w/t_0) - \frac{1}{2}]$, in which $E[X]$ is the largest integer not greater than X.

The expressions for $W_2^o(\omega)$ and $q_s(\omega)$ can be deduced as

$$W_2^o(\omega) = -2 \sum_{i=0}^{M_w} \mathrm{e}^{(2i+1)\mathrm{j}\tilde{k}x_0}(-1)^i = -2\mathrm{e}^{\mathrm{j}\tilde{k}x_0}\,\frac{1-\mathrm{e}^{2(M_w+1)\mathrm{j}\tilde{k}x_0}(-1)^{M_w+1}}{1+\mathrm{e}^{2\mathrm{j}\tilde{k}x_0}}$$

$$= \frac{[1-\mathrm{e}^{2(M_w+1)\mathrm{j}\tilde{k}x_0}(-1)^{M_w+1}]}{\cos(\tilde{k}x_0)}, \tag{24}$$

$$q_s(\omega) = -q_p(\omega)[1-\mathrm{e}^{2(M_w+1)\mathrm{j}\tilde{k}x_0}(-1)^{M_w+1}]/\cos(\tilde{k}x_0). \tag{25}$$

The modulus of the residual error signal $E(\omega)$ with an omnidirectional sensor is

$$|E(\omega)| = |p(x,\omega)| = \left|\frac{q_p(\omega)}{S}Z_p(x,\omega) + \frac{q_s(\omega)}{S}Z_s(x,\omega)\right| = \frac{|q_p(\omega)|}{S}|Z_p(x,\omega)|\mathrm{e}^{-2(M_w+1)\alpha x_0}. \tag{26}$$

If $M_w \gg 1$ then $M_w + 1 \approx \frac{1}{2}C_0 t_w/x_0$ and equation (26) becomes

$$|E(\omega)| = \{|q_p(\omega)|/S\}|Z_p(x,\omega)|\mathrm{e}^{-C_0 t_w \alpha}. \tag{27}$$

The active attenuation is now defined as the modulus of the ratio of the error signal without control to the error signal with control. Its expression in decibels is

$$\gamma(\omega) = 20\log_{10}[|q_p(\omega)||Z_p(x,\omega)|/S|E(\omega)|] = 8{\cdot}7 \times C_0 t_w \alpha \text{ (dB)}. \tag{28}$$

The expression for the active attenuation shows that it is independent of the angular frequency ω and of the location x of the error sensor. It is independent of the location x_0 of the secondary source as well. Equation (28) shows that the active attenuation can be improved by two means: a larger loss coefficient α or a longer time length t_w.

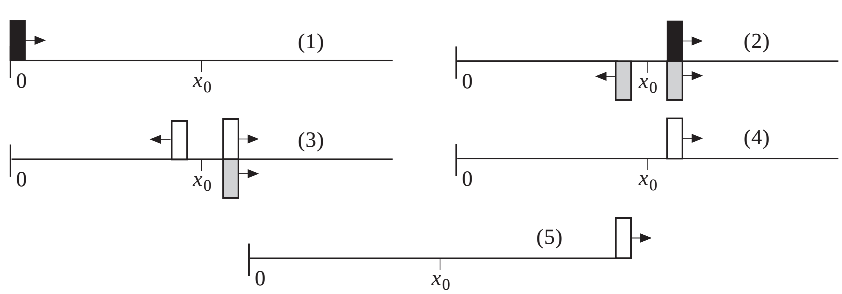

Figure 3. Principle of control with the approximate solution ($M_w = 1$, $\alpha = 0$).

The principle of control by the approximate weighting function W_2^o is explained by an example in Figure 3, where M_w and α are chosen equal to 1 and 0 respectively. As is shown in Figure 3, when a primary wave (in black) passes the secondary source (2), this secondary source emits a wave of the same amplitude but opposite sign in order to cancel out the primary excitation. At the same time a wave is also emitted upstream. It is reflected at the upstream termination and cancelled again by the control of the secondary source (3). This process continues as long as the weighting function of the controller is non-zero. When the time after the first emission exceeds the time length t_w, the control is impossible and the wave escapes downstream (4).

In Figure 3, the loss coefficient is zero and the control is useless. The amplitude of the outgoing wave is indeed equal to the amplitude of the ingoing primary wave (see (1) and (5)). Practically, the active attenuation is positive thanks to the positive loss coefficient. The attenuation comes indeed from the loss over the acoustic path covered by the wave. Since this acoustic path is potentially increased by the control, an active attenuation is possible.

3.2. OPTIMAL WEIGHTING FUNCTION OF THE CONTROLLER

The solution W_2^o that has been determined is approximate. One can now find the optimal solution W_2^* of the minimization problem.

The terms of the problem are as follows. The primary volume velocity $q_p(t)$ is now supposed to be a stationary random excitation of power spectral density $S_q(\omega)$. One is interested in the calculation of the expected value $\mathrm{E}[p^2(x, t)]$ of the squared sound pressure measured at the error sensor.

The expression for the sound pressure $p(x, t)$ can first be written as

$$p(x, t) = \frac{1}{S}\int_0^\infty q_p(t-\tau)Z_p(x, \tau)\,\mathrm{d}\tau + \frac{1}{S}\int_0^\infty q_s(t-\tau)Z_s(x, \tau)\,\mathrm{d}\tau. \qquad (29)$$

The weighting function $W(t)$ of the controller is the variable of the problem of minimization. This weighting function filters the reference signal, here equal to the volume velocity $q_p(t)$, to give the volume velocity $q_s(t)$ of the secondary source:

$$q_s(t) = \int_0^{t_w} q_p(t-\tau)W(\tau)\,\mathrm{d}\tau. \qquad (30)$$

Equations (29) and (30) give

$$p(x, y) = \frac{1}{S}\int_0^\infty q_p(t-\tau)Z_p(x, \tau)\,\mathrm{d}\tau$$
$$+ \int_0^{t_w}\left[\frac{1}{S}\int_0^\infty q_p(t-\tau_1-\tau_2)Z_s(x, \tau_1)\,\mathrm{d}\tau_1\right]W(\tau_2)\,\mathrm{d}\tau_2. \qquad (31)$$

Upon defining the functions

$$d(t) = \frac{1}{S}\int_0^\infty q_p(t-\tau)Z_p(x, \tau)\,\mathrm{d}\tau, \qquad k(t) = \frac{1}{S}\int_0^\infty q_p(t-\tau)Z_s(x, \tau)\,\mathrm{d}\tau, \qquad (32)$$

equation (31) can be rewritten as

$$p(x,t) = d(t) + \int_0^{t_w} k(t-\tau)W(\tau)\,\mathrm{d}\tau. \tag{33}$$

The minimization problem, whose solution will be denoted by W_2^* can be written as

$$\min_W J(W) = \mathrm{E}[p^2(x,t)] = \mathrm{E}\left[\left(d(t) + \int_0^{t_w} k(t-\tau)W(\tau)\,\mathrm{d}\tau\right)^2\right]. \tag{34}$$

The function $J(W)$ is a quadratic form,

$$J(W) = c + 2\int_0^{t_w} W(\tau)b(\tau)\,\mathrm{d}\tau + \int_0^{t_w}\int_0^{t_w} W(\tau_2)a(\tau_2-\tau_1)W(\tau_1)\,\mathrm{d}\tau_1\,\mathrm{d}\tau_2, \tag{35}$$

where

$$c = \mathrm{E}[d^2(t)] = \frac{1}{S^2}\frac{1}{2\pi}\int_{-\infty}^{\infty} |Z_p(x,\omega)|^2 S_q(\omega)\,\mathrm{d}\omega,$$

$$b(\tau) = \mathrm{E}[k(t-\tau)d(t)] = \frac{1}{S^2}\frac{1}{2\pi}\int_{-\infty}^{\infty} Z_p(x,\omega)\bar{Z}_s(x,\omega)S_q(\omega)\,\mathrm{e}^{-\mathrm{j}\omega\tau}\,\mathrm{d}\omega,$$

$$a(\tau) = \mathrm{E}[k(t)k(t+\tau)] = \frac{1}{S^2}\frac{1}{2\pi}\int_{-\infty}^{\infty} |Z_s(x,\omega)|^2 S_q(\omega)\,\mathrm{e}^{-\mathrm{j}\omega\tau}\,\mathrm{d}\omega. \tag{36}$$

The weighting function $W_2^*(t)$ that minimizes J satisfies the Wiener–Hopf integral equation

$$b(t) + \int_0^{t_w} a(t-\tau)W_2^*(\tau)\,\mathrm{d}\tau = 0, \qquad \forall t\in[0, t_w]. \tag{37}$$

If the stationary random signal is a white noise, the power spectral density $S_q(\omega)$ is independent of frequency, denoted by $S_q(\omega) = A_p$.

Now suppose in the rest of the calculation that the loss coefficient α is small with respect to $1/\lambda$. In this case, the primary and secondary paths have the simple forms

$$Z_p(x,\omega) = \rho_0 C_0\,\mathrm{e}^{j\tilde{k}x}, \qquad Z_s(x,\omega) = \rho_0 C_0\,\mathrm{e}^{\mathrm{j}\tilde{k}x}\cos(\tilde{k}x_0). \tag{38}$$

One can now calculate $b(\tau)$ and $a(\tau)$ as

$$b(\tau) = A_p\left(\frac{\rho_0 C_0}{S}\right)^2\left[\frac{1}{2\pi}\int_{-\infty}^{\infty}\cos(\bar{\tilde{k}}x_0)\,\mathrm{e}^{-\mathrm{j}\omega\tau}\,\mathrm{d}\omega\right]\mathrm{e}^{-2\alpha x}$$

$$= A_p\left(\frac{\rho_0 C_0}{S}\right)^2\left[\frac{\mathrm{e}^{\alpha x_0}}{2}\,\delta\left(\tau - \frac{x_0}{C_0}\right) + \frac{\mathrm{e}^{-\alpha x_0}}{2}\,\delta\left(\tau + \frac{x_0}{C_0}\right)\right]\mathrm{e}^{-2\alpha x},$$

$$a(\tau) = A_p\left(\frac{\rho_0 C_0}{S}\right)^2\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} |\cos(\tilde{k}x_0)|^2 \, \mathrm{e}^{-\mathrm{j}\omega\tau} \, \mathrm{d}\omega\right]\mathrm{e}^{-2\alpha x}$$

$$= A_p\left(\frac{\rho_0 C_0}{S}\right)^2\left[\frac{\mathrm{ch}\,(2\alpha x_0)}{2}\,\delta(\tau) + \frac{1}{4}\,\delta\left(\tau - \frac{2x_0}{C_0}\right) + \frac{1}{4}\,\delta\left(\tau + \frac{2x_0}{C_0}\right)\right]\mathrm{e}^{-2\alpha x}. \qquad (39)$$

Equations (37) and (39) give

$$(\mathrm{e}^{\alpha x_0}/2)\,\delta(t - t_0) + (\mathrm{ch}\,(2\alpha x_0)/2) W_2^*(t)\mathbf{1}_{[0,\, t_w]}(t) + \tfrac{1}{4} W_2^*(t - 2t_0) 1_{[2t_0,\, t_w]}(t)$$
$$+ \tfrac{1}{4}\mathrm{W}_2^*(\mathrm{t} + 2\mathrm{t}_0)\mathbf{1}_{[0,\, \mathrm{t_w} - 2\mathrm{t_0}]}(\mathrm{t}) = 0, \qquad \forall t \in [0, t_w], \qquad (40)$$

where $\mathbf{1}_{[a,\,b]}(t)$ is the characteristic function equal to one if $t \in [a, b]$ and equal to zero elsewhere.

$W_2^*(t)$ is found in the form

$$W_2^*(t) = \sum_{i=0}^{M_w} a_i \,\delta(t - (2i + 1)t_0). \qquad (41)$$

The coefficients a_i satisfy the following system of equations

$$\begin{array}{ccccccc}
-e^{\alpha x_0}/2 & = & & & [\mathrm{ch}\,(2\alpha x_0)/2]a_0 & + & \tfrac{1}{4}a_1, \\
0 & = & \tfrac{1}{4}a_0 & + & [\mathrm{ch}\,(2\alpha x_0)/2]a_1 & + & \tfrac{1}{4}a_2, \\
\vdots & = & \vdots & & \vdots & & \vdots \\
0 & = & \tfrac{1}{4}a_i & + & [\mathrm{ch}\,(2\alpha x_0)/2]a_{i+1} & + & \tfrac{1}{4}a_{i+2}, \\
\vdots & = & \vdots & & \vdots & & \vdots \\
0 & = & \tfrac{1}{4}a_{M_w - 2} & + & [\mathrm{ch}\,(2\alpha x_0)/2]a_{M_w - 1} & + & \tfrac{1}{4}a_{M_w} \\
0 & = & \tfrac{1}{4}a_{M_w - 1} & + & [\mathrm{ch}\,(2\alpha x_0)/2]a_{M_w}. & &
\end{array} \qquad (42)$$

The coefficients a_i satisfy a recursion series,

$$a_i = -2\mathrm{ch}\,(2\alpha x_0)a_{i+1} - a_{i+2}, \qquad (43)$$

with

$$a_{M_w - 1} = -2\mathrm{ch}\,(2\alpha x_0)a_{M_w}. \qquad (44)$$

The general coefficient $a_{M_w - i}$ takes the form

$$a_{M_w - i} = (-1)^i[A\,\mathrm{e}^{2i\alpha x_0} + B\,\mathrm{e}^{-2i\alpha x_0}] \qquad (45)$$

with

$$A = a_{M_w}\,\mathrm{e}^{2\alpha x_0}/2\mathrm{sh}\,(2\alpha x_0), \qquad B = -a_{M_w}\,\mathrm{e}^{-2\alpha x_0}/2\mathrm{sh}\,(2\alpha x_0). \qquad (46)$$

The series has the expression:

$$a_{M_w - i} = (-1)^i\,\frac{\mathrm{sh}\,[2(i+1)\alpha x_0]}{\mathrm{sh}\,[2\alpha x_0]}\,a_{M_w}. \qquad (47)$$

The first equality of the system (42) gives also

$$-\frac{\mathrm{e}^{\alpha x_0}}{2} = (-1)^{M_w}\,\frac{\mathrm{sh}\,[2(M_w + 2)\alpha x_0]}{4\mathrm{sh}\,[2\alpha x_0]}\,a_{M_w}\,. \qquad (48)$$

The two equations (47) and (48) lead to:

$$a_{M_w - i} = 2(-1)^{M_w - i + 1} \frac{\mathrm{sh}\,[2(i+1)\alpha x_0]}{\mathrm{sh}\,[2(M_w + 2)\alpha x_0]} \mathrm{e}^{\alpha x_0}. \tag{49}$$

The following formula is then deduced:

$$a_i = 2(-1)^{i+1} \frac{\mathrm{sh}\,[2(M_w - i + 1)\alpha x_0]}{\mathrm{sh}\,[2(M_w + 2)\alpha x_0]} \mathrm{e}^{\alpha x_0}. \tag{50}$$

Consider now the following borderline cases.

Infinite time length $M_w \to \infty$. The relevant formula for $W_2(t)$ is that when there is no constraint on the time length of the weighting function. The expression for a_i is then found to be

$$a_i = 2(-1)^{i+1}\,\mathrm{e}^{-(2i+1)\alpha x_0}. \tag{51}$$

Note here the coefficients for the expression for $W_2(t)$ in equation (19).

Waveguide with no loss, $\alpha \to 0$. This case corresponds to a waveguide with perfectly rigid walls and no boundary layer attenuation. One can compare the results for $W_2^o(t)$ and $W_2^*(t)$: for $W_2^o(t)$

$$a_i = 2(-1)^{i+1}; \tag{52}$$

for $W_2^*(t)$,

$$a_i = 2(-1)^{i+1}(M_w - i + 1)/(M_w + 2). \tag{53}$$

The principle of control with the optimal solution is presented in Figure 4 with M_w and α equal to 1 and 0 respectively. To simplify the problem, consider a primary excitation of amplitude 1 (1). A secondary wave of amplitude $-2/3$ is emitted (2) in order to control partially the primary wave. A secondary wave of same amplitude is emitted also upstream and is reflected at the upstream termination. When this wave passes the secondary source (3), this source emits a wave of amplitude 1/3 and controls partially the outgoing wave. A secondary wave of amplitude 1/3 is emitted also upstream and is reflected at the upstream termination. It escapes then downstream (5) because the time length of the weighting function is exceeded.

With the optimal controller, there are three outgoing waves of amplitude 1/3 or $-1/3$. The results in terms of energy (the square of the amplitude) give an energy equal to $3[1/3]^2 = 1/3$.

With the approximate controller, there is a single outgoing wave of amplitude 1: i.e., an energy equal to 1.

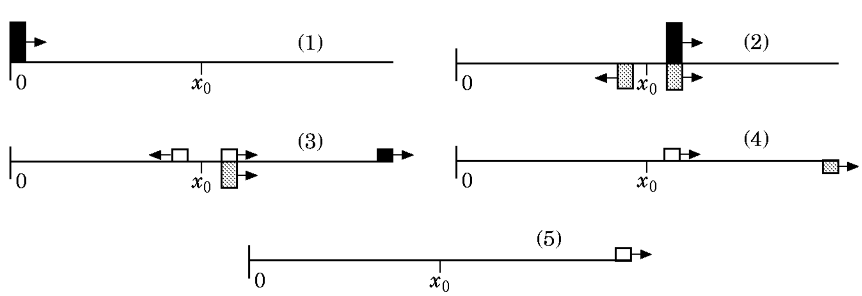

Figure 4. Principle of control with the optimal solution ($M_w = 1$, $\alpha = 0$).

It is verified that the control with the optimal controller is more efficient than the control with the approximate weighting function first proposed.

3.3. OPTIMAL TRANSFER FUNCTION OF THE CONTROLLER

In the previous section, the expression for the optimal weighting function $W_2^*(t)$ has been determined as

$$W_2^*(t) = -2^{\alpha x_0} \sum_{i=0}^{M_w} (-1)^i \frac{\mathrm{sh}\,[2(M_w - i + 1)\alpha x_0]}{\mathrm{sh}\,[2(M_w + 2)\alpha x_0]} \delta(t - (2i+1)t_0). \tag{54}$$

In order to evaluate the active attenuation with respect to frequency, it is necessary to calculate the Fourier transform of $W_2^*(t)$. The details of the calculation of the transfer function $W_2^*(\omega)$ are presented in Appendix B. Its expression is

$$W_2^*(\omega) = -\frac{\mathrm{sh}\,[2(M_w+1)\alpha x_0] + \mathrm{e}^{2\mathrm{j}kx_0}\mathrm{sh}\,[2(M_w+2)\alpha x_0] + \mathrm{e}^{2(M_w+2)\mathrm{j}kx_0}(-1)^{M_w}\mathrm{sh}\,[2\alpha x_0]}{(1 + \mathrm{e}^{2\mathrm{j}\tilde{k}x_0})\cos(\tilde{\tilde{k}}x_0)\mathrm{sh}\,[2(M_w+2)\alpha x_0]}. \tag{55}$$

The power spectral density of the sound pressure $p(x, t)$ is

$$\begin{aligned} S_p(x, \omega) &= [S_q(\omega)/S^2]|Z_p(x, \omega) + W_2^*(\omega) Z_s(x, \omega)|^2 \\ &= (A_p/S^2)|Z_p(x, \omega)^2|1 + W_2^*(\omega)\cos(\tilde{k}x_0)|^2. \end{aligned} \tag{56}$$

The active attenuation $\gamma(\omega)$ is defined as the ratio of the power spectral density $S_p(x, \omega)$ without control to this power spectral density with control. Its expression in decibels is

$$\gamma(\omega) = 20 \log_{10}\left[\frac{1}{|1 + W_2^*(\omega)\cos(\tilde{k}x_0)|}\right]. \tag{57}$$

When $\alpha \to 0$ (i.e., waveguide without any loss), the expression for the attenuation takes the simplified form

$$\gamma(\omega) = 20 \log_{10}\left[\frac{2(M_w + 2)|\cos(kx_0)|}{|1 - (-1)^{M_w}\,\mathrm{e}^{2(M_w+2)\mathrm{j}kx_0}|}\right]. \tag{58}$$

The acoustic attenuation remains independent of the location x of the error microphone. Whereas the acoustic attenuation with the approximate controller W_2^o was also independent of the angular frequency ω and of the location x_0 of the secondary source, the acoustic attenuation with the optimal controller W_2^* depends now on the frequency and on the location of the secondary source. The minima of this attenuation are found at the frequencies f_m of section 2.

Equation (58) shows that no attenuation can be achieved at these frequencies for any time length t_w if the loss coefficient α is zero. When the loss coefficient α is positive, the acoustic attenuation is small at these frequencies f_m but can be increased with the time length t_w.

As was said in section 2, the frequencies f_m correspond also to a large transfer function of the controller. Evanescent modes can be largely excited and the active attenuation is limited. The work presented here points out that the truncation of the weighting function reduces as well the active attenuation in narrow bands of frequencies centered around these frequencies f_m.

Applications. Consider the dimensions a and x_0 equal 0·1 m and 0·5 m respectively, and the speed of sound equal to 344 m s^{-1}. The walls of the waveguide are supposed to be perfectly rigid. The losses come then mainly from the boundary layer attenuation. The expression of the loss coefficient due to the boundary layer attenuation is $2{\cdot}38 \times 10^{-5}\sqrt{\omega/a}$ m^{-1} [14]. Since a loss coefficient independent of frequency is being used, one chooses for α the loss coefficient at 500 Hz at the center of bandwidth (0–1000) Hz of the study. The value for α is therefore $1{\cdot}33 \times 10^{-2}$ m^{-1}. Figure 5 shows the attenuation $\gamma(x, \omega)$ as a function of frequency. The frequencies f_m are equal to $(2m + 1) \times 172$ Hz.

4. EFFECT OF SAMPLING RATE

Here the effect of the sampling rate on the active attenuation is considered. Let t_g denote the period between two samples. The sampling frequency is therefore equal to $1/t_g$.

When a continuous signal is sampled a problem arises when this signal contains frequencies which are higher than half the sampling frequency [5]. High frequencies are indeed indistinguishable from lower frequencies. This phenomenon is called aliasing. This is why these high frequencies are usually filtered out by an anti-aliasing filter in the analogue signal before its conversion into digital form. The ideal transfer function of this ideal anti-aliasing filter is equal to $\mathbf{1}_{[-\omega_g/2,\,\omega_g/2]}(\omega)$ where the angular frequency ω_g is equal to $2\pi/t_g$. In order to take into account the use of an anti-aliasing filter, a new transfer function $W_2^g(\omega)$ of the controller is introduced. Its expression is

$$W_2^g(\omega) = W_2^*(\omega)\mathbf{1}_{[-\omega_g/2,\,\omega_g/2]}(\omega). \quad (59)$$

The interpretation of equation (59) is that the use of anti-aliasing filters forbids any control at frequencies higher than half the sampling frequency.

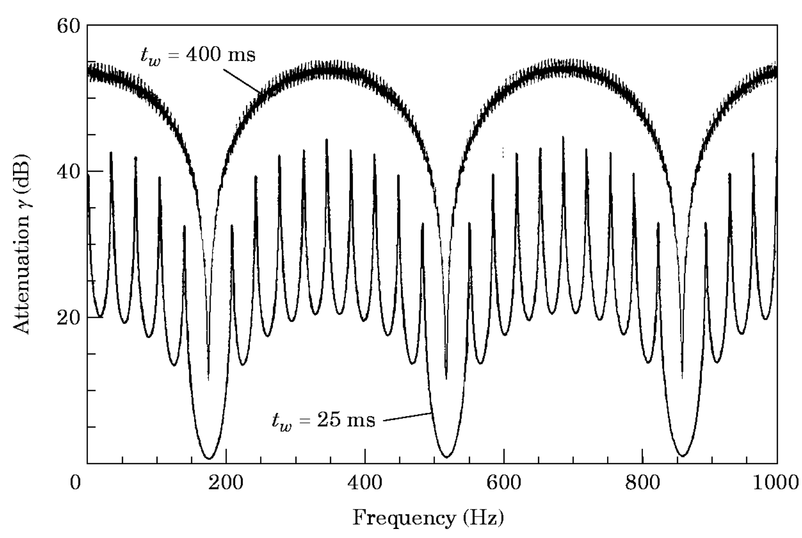

Figure 5. Acoustic attenuation $\gamma(\omega)$ with a time length t_w varying from 25 ms to 400 ms.

These two characteristic times (the time length of the weighting function t_w and the (sampling rate t_g) introduced previously, are actually not independent. In order to point out their dependence, one must introduce the maximum number f_{DSP} of multiplications that the processor can perform per second. $t_g f_{DSP}$ is the maximum number of multiplications performed by the processor during a sampling period.

For an adaptive algorithm like X-LMS, the number of multiplications required during a sampling period is equal to $N_s + 2N_w$ where N_s and N_w are the number of coefficients of the digital filters representing the weighting functions of the secondary path $Z_s(t)$ and the controller $W(t)$ respectively. The constraint coming from an *on-line* system has therefore the following expression:

$$N_s + 2N_w \leqslant t_g f_{DSP}. \tag{60}$$

The time length is equal to the number of coefficients of the digital filter multiplied by the sampling period t_g. If the time length used for the secondary path is noted t_s, the inequality (60) gives

$$t_s + 2t_w \leqslant t_g^2 f_{DSP}. \tag{61}$$

The objective of the next section is to optimize the parameters (t_w, t_g) of the control if the speed of the DSP is known. The lengths of the weighting functions are therefore maximized:

$$t_s + 2t_w \approx t_g^2 f_{DSP}. \tag{62}$$

In order to simplify the problem, equal time lengths will be considered, and therefore in the rest of this study the equation

$$3t_w \approx t_g^2 f_{DSP} \tag{63}$$

is used.

5. OPTIMIZATION OF SAMPLING RATE AND TRUNCATION

In this section, the primary signal is a white noise convolved with an ideal low pass filter of cut-off frequency $1/t_f$. This situation corresponds to practical cases in ventilation ducts where the primary excitation is mainly composed of low frequency components. The power spectral density $S_q(\omega)$ of the primary signal is now equal to $A_p \mathbf{1}_{[-\omega_f, \omega_f]}(\omega)$ where ω_f is equal to $2\pi/t_f$.

The power spectral density of the sound pressure $S_p(x, \omega)$ with the transfer function $W_2^g(\omega)$ of the controller, with the effects of truncation and sampling simultaneously taken into account, is

$$\begin{aligned} S_p(x, \omega) &= (S_q(\omega)/S^2)|Z_p(x, \omega) + W_2^g(\omega) Z_s(x, \omega)|^2 \\ &= (A_p/S^2)|Z_p(x, \omega)|^2 |1 + W_2^g(\omega) \cos(\tilde{k} x_0)|^2 \mathbf{1}_{[-\omega_f, \omega_f]}(\omega) \\ &= A_p(\rho_0 C_0/S)^2 \, \mathrm{e}^{-2\alpha x} |1 + W_2^g(\omega) \cos(\tilde{k} x_0)|^2 \mathbf{1}_{[-\omega_f, \omega_f]}(\omega). \end{aligned} \tag{64}$$

By using this expression for the power spectral density of the sound pressure, one finds

$J(W_g^2)$, representing the expected value of the squared sound pressure with the transfer function $W_2^g(\omega)$ of the controller:

$$J(W_2^g) = \mathrm{E}[p^2(x, t)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_p(x, \omega)\, \mathrm{d}\omega$$

$$= A_p\left(\frac{\rho_0 C_0}{S}\right)^2 \frac{\mathrm{e}^{-2\alpha x}}{2\pi}\int_{-\omega_f}^{\omega_f} [1 + W_2^g(\omega)\cos(\tilde{k}x_0)|^2\, \mathrm{d}\omega. \tag{65}$$

By introducing the expression for the transfer function $W_2^g(\omega)$ in equation (65), $J(W_2^g)$ can be determined (see Appendix C) as

$$J(W_2^g) = A_p\left(\frac{\rho_0 C_0}{S}\right)^2 \mathrm{e}^{-2\alpha x}\left\{\frac{2}{t_f} - \frac{2}{t_{\max}}\frac{\mathrm{sh}\,[2(M_w + 1)\alpha x_0]}{\mathrm{sh}\,2(M_w + 2)\alpha x_0]}\mathrm{e}^{2\alpha x_0}\right\}. \tag{66}$$

Here $t_{\max} = \max(t_f, 2t_g)$ and it has been supposed that the ratio $2t_0/t_{\max}$ is an integer, in order to simplify the expression for the objective function. If this ratio is not an integer but large enough, this expression gives a good estimate of $J(W_2^g)$.

The active attenuation γ can now be found. This is the ratio of the value of the objective function without control to its value with control. Its expression in decibels is

$$\gamma = 10\log_{10}[J(0)/J(W_2^g)] = -10\log_{10}\left[1 - \frac{t_f}{t_{\max}}\frac{\mathrm{sh}\,[2(M_w + 1)\alpha x_0]}{\mathrm{sh}\,[2(M_w + 2)\alpha x_0]}\mathrm{e}^{2\alpha x_0}\right]. \tag{67}$$

One can now introduce four non-dimensional numbers:

$$T_w = t_w/t_0 \text{ (time length)}, \qquad T_g = t_g/t_f \text{ (sampling period)};$$

$$\alpha_0 = \alpha x_0 \quad \text{(loss coefficient)}, \qquad V_{DSP} = t_f^2 f_{DSP}/t_0 \text{ (speed of the processor)}. \tag{68}$$

The active attenuation γ depends on three of them (T_g, T_w and α_0):

$$\gamma = -10\log_{10}[1 - G_g(T_g)G_w(T_w, \alpha_0)]. \tag{69}$$

$$G_g(u) = \frac{1}{\max(1, 2u)}, \qquad G_w(u, \alpha_0) = \mathrm{e}^{2\alpha_0}\frac{\mathrm{sh}\,\{2[M_w(u) + 1]\alpha_0\}}{\mathrm{sh}\,\{2[M_w(u) + 2]\alpha_0\}}, \qquad M_w(u) = E\left[\frac{u}{2} - \frac{1}{2}\right]. \tag{70}$$

Equation (69) shows clearly that the active attenuation γ depends separately on two factors: the non-dimensional sampling period T_g and the non-dimensional time length T_w of the weighting function. Their respective effects are summed up in the variations of the functions G_g and G_w. These variations are presented in Figure 6. On the one hand the function G_g decreases from 1 to 0 when the non-dimensional sampling period increases. On the other hand the function G_w increases from 0 to 1 when the non-dimensional time length of the controller increases. Whereas G_g is independent of the non-dimensional loss coefficient α_0, G_w is all the larger since this loss coefficient is large. It is known indeed that a large loss coefficient reduces the detrimental effect of the upstream reflection. At first sight, the optimal set-up point consists of a small non-dimensional sampling period T_g and a long non-dimensional time length T_w. Unfortunately, when the active system is constrained to act *on-line*, equation (63) must be satisfied and it forbids this ideal set-up point.

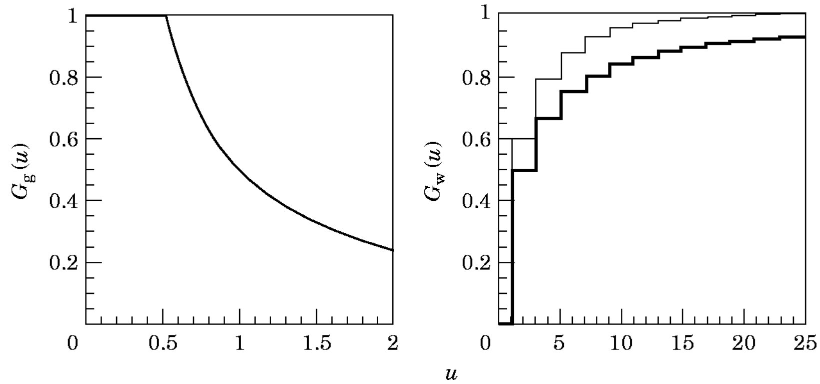

Figure 6. Variations of the functions $G_g(u)$ and $G_w(u)$. ——; $\alpha_0=0$; ——; $\alpha_0=0{\cdot}1$ (a) $V_{DSP}=50$; (b) $V_{DSP}=5$.

Equation (63) can be rewritten, however, as a function of the non-dimensional numbers,

$$3T_w = T_g^2 V_{DSP}. \tag{71}$$

For any non-dimensional speed V_{DSP} of the DSP and any non-dimensional loss coefficient α_0, there is however an optimal set-up point in the plane $(T_w, 1/T_g)$. Using the constraint (71) for an *on-line* system, γ can be expressed as a function of T_g only. The acoustic attenuation γ can then be maximized with respect to the non-dimensional sampling period T_g. For any V_{DSP} and α_0, a maximum can be determined with the corresponding solution for T_g (see Figure 7).

Two types of configurations can be identified as functions of the value of V_{DSP}.

The first type occurs when the non-dimensional speed of the processor V_{DSP} is greater than 12 (see the first example of Figure 7). In this case the acoustic attenuation γ increases with the non-dimensional sampling period T_g in the segment $[0, 0{\cdot}5]$ because $G_g(T_g)$ is constant and $G_w(\frac{1}{3}T_g^2 V_{DSP}, \alpha_0)$ increases with T_g. Since $V_{DSP} > 12$, when T_g is equal to $0{\cdot}5$, the non-dimensional time length T_w is greater than 1 and the acoustic attenuation γ is strictly positive. For T_g greater than $0{\cdot}5$ approximately, it is demonstrated in Appendix D that the acoustic attenuation decreases with T_g.

The second type of configuration occurs when the non-dimensional speed of the processor V_{DSP} is less than 12 (see the second example of Figure 7). In this case there is no positive acoustic attenuation until T_g is equal $\sqrt{3/V_{DSP}}$ (i.e., the non-dimensional time length T_w is equal to 1). For T_g greater than $\sqrt{3/V_{VSP}}$ it can be demonstrated, as in the previous case, that the acoustic attenuation decreases with T_g.

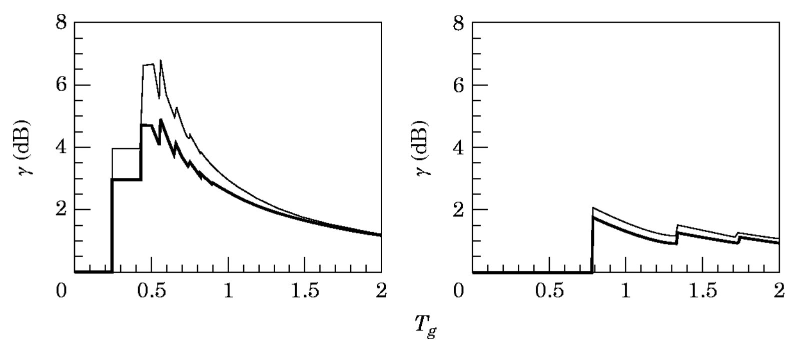

Figure 7. Acoustic attenuation γ with respect to the non-dimensional time length T_g. Key as Figure 6.

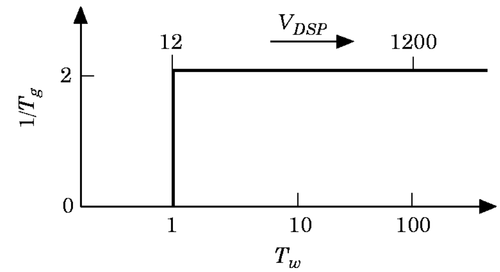

Figure 8. Optimal non-dimensional numbers (T_w, $1/T_g$) as functions of V_{DSP}.

Figure 8 sums up the results for the variations of the optimal parameters of control in the plane (T_w, $1/T_g$) as a function of V_{DSP}. For small values of V_{DSP}, the non-dimensional time length T_w must be fixed to 1 and the non-dimensional sampling rate increases with the non-dimensional speed of the processor. A transition occurs when V_{DSP} is equal to 12. This value corresponds to the point (1, 2) in the plane (T_w, $1/T_g$). When the non-dimensional sampling frequency is high enough (i.e., it is equal to 2), it becomes useless to increases it more. The control is then improved by lengthening the non-dimensional time T_w. It should be underlined that most of the applications of active control concern a non-dimensional speed V_{DSP} that is greater than the critical value equal to 12.

One can now examine the influence of the non-dimensional loss coefficient α_0. It is found on the one hand that the presence of loss does not modify the value of the optimal parameter T_g. On the other hand, the active attenuation is improved by the presence of loss.

The acoustic attenuation γ can now be defined for the optimal couple parameters ($1/T_g$, T_w) determined previously:

$$\gamma = -10 \log_{10} [1 - G_g(\sqrt{3/V_{DSP}})G_w(1, \alpha_0)], \quad V_{DSP} \leqslant 12,$$

$$\gamma = -10 \log_{10} [1 - G_w(V_{DSP}/12, \alpha_0)], \quad V_{DSP} \geqslant 12. \tag{72}$$

It is noticeable that this attenuation depends now on only two non-dimensional parameters, V_{DSP} and α_0. The variations of γ with respect to the non-dimensional speed of the processor V_{DSP} are presented in Figure 9 for α_0 equal to 0 and 0·1 respectively. These curves confirm that the two parameters V_{DSP} and α_0 must be chosen as large as possible in order to get the maximum acoustic attenuation. Consider now the asymptotic variations of γ when the parameter V_{DSP} is large. Note now that $M_w = M_w(V_{DSP}/12)$ and $V_{DSP} > 12$. One has

$$\gamma = -10 \log_{10} \left[(1 - e^{-4\alpha_0}) \frac{e^{-4\alpha_0(M_w + 1)}}{1 - e^{-4\alpha_0(M_w + 2)}} \right]. \tag{73}$$

Suppose that $V_{DSP} \gg 12$. This leads to

$$\gamma = -10 \log_{10} \left[(1 - e^{-4\alpha_0}) \frac{e^{-\frac{1}{6}\alpha_0 V_{DSP}}}{1 - e^{-\frac{1}{6}\alpha_0 V_{DSP}}} \right]. \tag{74}$$

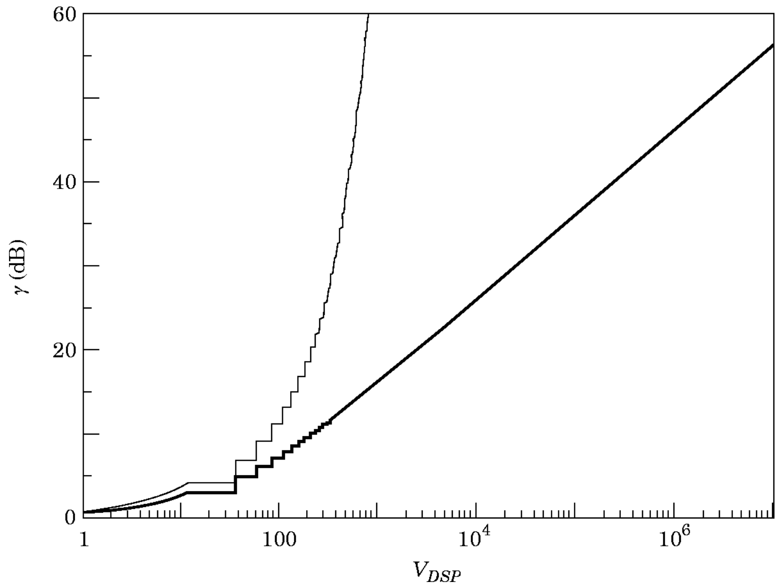

Figure 9. Acoustic attenuation γ as a function of V_{DSP}. Key as Figure 6.

Consider now two cases of the values of the product $\alpha_0 V_{DSP}$:

$$\gamma = 10\log_{10}[V_{DSP}/24], \quad \alpha_0 V_{DSP} \ll 6;$$

$$\gamma = 0{\cdot}72\alpha_0 V_{DSP} - 10\log_{10}[1 - e^{-4\alpha_0}], \quad \alpha_0 V_{DSP} \gg 6. \tag{75}$$

The asymptotic variations of γ when the parameter V_{DSP} is large are then

$$\gamma \sim 10\log_{10}(V_{DSP}), \quad \alpha_0 = 0; \qquad \gamma \sim 0{\cdot}72\alpha_0 V_{DSP}, \quad \alpha_0 > 0. \qquad (76, 77)$$

It is noticed that the presence of loss changes radically the asymptotic behaviour of the system. Without loss the non-dimensional speed of the processor must be increased by a factor of 10 to get 10 dB of additional attenuation. With the presence of loss, the number of decibels of attenuation increases linearly with V_{DSP}.

The two non-dimensional parameters V_{DSP} and α_0 themselves depend on four parameters: α, t_f, f_{DSP} and x_0 (or t_0). One can comment on their respective influence on the active attenuation as follows.

The loss coefficient α enters into the expression of α_0. This loss coefficient should be increased with the help of absorbent materials for example. Losses reduce the detrimental effect of the upstream reflection.

The characteristic time t_f of the primary signal enters into the expression for V_{DSP}. Its influence is very important since V_{DSP} depends on this squared time. The acoustic attenuation will be better if the primary signal is limited to low frequencies.

The speed of the processor f_{DSP} enters also into the expression for V_{DSP}. Faster processors are required to reach higher levels of active attenuation.

The location of the secondary source enters simultaneously into the expression of V_{DSP} and α_0.

The optimal location for the secondary source is found at the location of the primary source. α_0 is then equal to zero and V_{DSP} is infinite. The asymptotic expression (76) is then valid and the active attenuation is infinite. Unfortunately, the constraint of causality combined with electrical delays imposes a minimum distance x_0 between the primary and the secondary source. For a waveguide without loss, α_0 is again equal to zero and the active attenuation depends only on the non-dimensional parameter V_{DSP}. This leads to the fact that the secondary source should be located as near as possible to the primary source in order to maximize V_{DSP} and the active attenuation γ. For a lossy waveguide, the placement of the secondary source is not as obvious as previously. An increase of x_0 leads indeed to two changes of the non-dimensional parameters (reduction V_{DSP} and increase of α_0) that have contrary effects on the active attenuation. Consider the product $\alpha_0 V_{DSP}$ of equations (75). The non-dimensional parameters are replaced by the real quantities that compose them:

$$\alpha_0 V_{DSP} = \alpha C_0 t_f^2 f_{DSP}. \tag{78}$$

Equation (78) shows that this product is independent of the location of the source x_0. One can now distinguish two different behaviours as functions of the product $\alpha_0 V_{DSP}$. If $\alpha_0 V_{DSP} \ll 6$ then the active attenuation depends only on the non-dimensional parameter V_{DSP}. This leads to the fact that the secondary source should be located as near as possible to the primary source in order to maximize V_{DSP} and the active attenuation γ. This case is similar to a non-lossy waveguide. If $\alpha_0 V_{DSP} \gg 6$ then the active attenuation depends only on the non-dimensional parameter α_0 and on the product $\alpha_0 V_{DSP}$. Since the latter product is independent of the location of the secondary source, the secondary source should be located as near as possible to the primary source in order to minimize α_0 and maximize the active attenuation γ. It should be underlined that the term $-10 \log_{10} [1 - e^{-4\alpha_0}]$ is less than 2 dB when $\alpha_0 > \frac{1}{4}$. This means that all the locations whose co-ordinate x_0 is greater than $1/4\alpha$ can be considered equivalent in terms of active attenuation.

6. CONCLUSIONS

Three active noise control systems for a plane sound wave in a finite lossy waveguide have been reviewed: the active system with feedback and omnidirectional sensors; the active system with feedback and unidirectional sensors; the active system with an independent reference signal. For the first and third systems, the controller has a long time response. The truncation of the weighting function can therefore greatly reduce the active attenuation. When the active control system is constrained to act *on-line*, the optimal truncated weighting function has been determined for an independent reference signal. It has been pointed out that a controller, acting with this weighting function, provides an active attenuation that is limited in narrow bands of frequencies. These frequencies correspond to a large transfer function of the controller.

The active attenuation with the combined effects of sampling and truncation has also been determined. The sampling imposes the use of anti-aliasing filters and no attenuation can be achieved at frequencies greater than half the sampling frequency.

The optimal set-up point of the time length and of the sampling rate of the controller has been found. It is shown that the active attenuation depends on two non-dimensional parameters if the system is optimized. These non-dimensional parameters themselves depend on four practical parameters, which are the characteristic time of the primary signal, the speed of the processor, the loss coefficient in the waveguide and the location of the secondary source.

It has been shown that the secondary source should be located as near as possible to the primary source for a lossy or a non-lossy waveguide. For a lossy waveguide and when the speed of the processor is large, all the locations of the secondary source beyond a critical point are equivalent in terms of active attenuation.

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APPENDIX A: DETERMINATION OF THE SOUND PRESSURE FIELD

To find the solution $p(x, \omega)$ of the differential equations (2), one can first write $p(x, \omega)$ as

$$p(x, \omega) = B^+ \mathrm{e}^{\mathrm{j}\tilde{k}x} + B^- \mathrm{e}^{-\mathrm{j}\tilde{k}x} \quad \forall x \geqslant x_0,$$

$$p(x, \omega) = A^+ \mathrm{e}^{\mathrm{j}\tilde{k}x} + A^- \mathrm{e}^{-\mathrm{j}\tilde{k}x} \quad \forall x \leqslant x_0, \tag{A1}$$

where A^+, A^-, B^+ and B^- are constants to be determined by the boundary conditions. The boundary condition at $x = 0$ gives

$$A^+ - A^- = \frac{\rho_0 C_0}{1 + \mathrm{j}C_0\alpha/\omega}\,\frac{q_p(\omega)}{S}\,. \tag{A2}$$

The boundary condition at $x = \infty$ gives

$$B^- = 0. \tag{A3}$$

The continuity of sound pressure at $x = x_0$ gives

$$B^+ \mathrm{e}^{\mathrm{j}\tilde{k}x_0} = A^+ \mathrm{e}^{\mathrm{j}\tilde{k}x_0} + A^- \mathrm{e}^{-\mathrm{j}\tilde{k}x_0}. \tag{A4}$$

The discontinuity of the first derivative of sound pressure at $x = x_0$ gives

$$B^+ \mathrm{e}^{\mathrm{j}\tilde{k}x_0} - A^+ \mathrm{e}^{\mathrm{j}\tilde{k}x_0} + A^- \mathrm{e}^{-\mathrm{j}\tilde{k}x_0} = \frac{\rho_0 C_0}{1 + \mathrm{j}C_0\alpha/\omega}\,\frac{q_s(\omega)}{S}\,. \tag{A5}$$

The solution of the system of equations (A2)–(A5) gives, for q_s equal to zero,

$$A^+ = B^+ = \frac{\rho_0 C_0}{1 + \mathrm{j}C_0\alpha/\omega}\frac{q_p(\omega)}{S}, \qquad A^- = B^- = 0. \tag{A6}$$

The solution of the system of equations (A2)–(A5) gives, for q_p equal to zero,

$$B^+ = \frac{\rho_0 C_0}{1 + \mathrm{j}C_0\alpha/\omega}\frac{q_s(\omega)}{S}\cos(\tilde{k}x_0), \qquad B^- = 0, \qquad A^+ = A^- = \frac{\mathrm{e}^{\mathrm{j}\tilde{k}x_0}}{2\cos(\tilde{k}x_0)}B^+. \tag{A7}$$

These results lead to the solution

$$p(x,\omega) = [q_p(\omega)/S]Z_p(x,\omega) + [q_s(\omega)/S]Z_s(x,\omega),$$

$$Z_p(x,\omega) = \frac{\rho_0 C_0}{1 + \mathrm{j}C_0\alpha/\omega}\mathrm{e}^{\mathrm{j}\tilde{k}x}\ \forall x, \qquad Z_s(x,\omega) = Z_p(x,\omega)\cos(\tilde{k}x_0)\ \forall x \geqslant x_0,$$

$$Z_s(x,\omega) = Z_p(x_0,\omega)\cos(\tilde{k}x)\ \forall x \leqslant x_0. \tag{A8}$$

APPENDIX B: CALCULATION OF $W_2^*(\omega)$

The optimal weighting function $W_2^*(t)$ has the expression

$$W_2^*(t) = -2\mathrm{e}^{\alpha x_0}\sum_{m=0}^{M_w}(-1)^m\frac{\mathrm{sh}\,[2(M_w - m + 1)\alpha x_0]}{\mathrm{sh}\,[2(M_w + 2)\alpha x_0]}\delta(t - (2m+1)t_0). \tag{B1}$$

The Fourier transform of $W_2^*(t)$ is

$$W_2^*(\omega) = -2\frac{\mathrm{e}^{\mathrm{j}\bar{\tilde{k}}x_0}}{\mathrm{sh}\,[2(M_w + 2)\alpha x_0]}\sum_{m=0}^{M_w}(-1)^m\,\mathrm{sh}\,[2(M_w - m + 1)\alpha x_0]\,\mathrm{e}^{2m\mathrm{j}kx_0}$$

$$= -\frac{\mathrm{e}^{\mathrm{j}\bar{\tilde{k}}x_0}}{\mathrm{sh}\,[2(M_w + 2)\alpha x_0]}\sum_{m=0}^{M_w}(-1)^m[\mathrm{e}^{2(M_w+1)\alpha x_0}\,\mathrm{e}^{2m\mathrm{j}\tilde{k}x_0} - \mathrm{e}^{-2(M_w+1)\alpha x_0}\,\mathrm{e}^{2m\mathrm{j}\bar{\tilde{k}}x_0}]$$

$$= -\frac{\mathrm{e}^{\mathrm{j}\bar{\tilde{k}}x_0}}{\mathrm{sh}\,[2(M_w + 2)\alpha x_0]}\left\{\mathrm{e}^{2(M_w+1)\alpha x_0}\frac{1 + (-1)^{M_w}\,\mathrm{e}^{2(M_w+1)\mathrm{j}\tilde{k}x_0}}{1 + \mathrm{e}^{2\mathrm{j}\tilde{k}x_0}}\right.$$

$$\left. - \mathrm{e}^{-2(M_w+1)\alpha x_0}\frac{1 + (-1)^{M_w}\,\mathrm{e}^{2(M_w+1)\mathrm{j}\bar{\tilde{k}}x_0}}{1 + \mathrm{e}^{2\mathrm{j}\bar{\tilde{k}}x_0}}\right\}$$

$$= -\frac{2\,\mathrm{e}^{\mathrm{j}\bar{\tilde{k}}x_0}}{\mathrm{sh}\,[2(M_w + 2)\alpha x_0](1 + \mathrm{e}^{2\mathrm{j}\tilde{k}x_0})(1 + \mathrm{e}^{2\mathrm{j}\bar{\tilde{k}}x_0})}\{\mathrm{sh}\,[2(M_w + 1)\alpha x_0]$$

$$+ \mathrm{e}^{2\mathrm{j}kx_0}\,\mathrm{sh}\,[2(M_w + 2)\alpha x_0] + \mathrm{e}^{2(M_w+2)\mathrm{j}kx_0}(-1)^{M_w}\,\mathrm{sh}\,[2\alpha x_0]\}, \tag{B2}$$

or

$$W_2^*(\omega) = -\frac{\mathrm{sh}\,[2(M_w+1)\alpha x_0] + \mathrm{e}^{2\mathrm{j}kx_0}\,\mathrm{sh}\,[2(M_w+2)\alpha x_0] + \mathrm{e}^{2(M_w+2)\mathrm{j}kx_0}(-1)^{M_w}\,\mathrm{sh}\,[2\alpha x_0]}{(1+\mathrm{e}^{2\mathrm{j}\tilde{k}x_0})\cos(\tilde{\bar{k}}x_0)\,\mathrm{sh}\,[2(M_w+2)\alpha x_0]}. \tag{B3}$$

APPENDIX C: CALCULATION OF $J(W_2^g)$

$J(W_2^g)$, which represents the expected value of the squared sound pressure with the transfer function $W_2^g(\omega)$ of the controller is given by

$$J(W_2^g) = \mathrm{E}[p^2(x,t)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_p(x,\omega)\,\mathrm{d}\omega$$

$$= A_p\left(\frac{\rho_0 C_0}{S}\right)^2 \frac{\mathrm{e}^{-2\alpha x}}{2\pi}\int_{-\omega_f}^{\omega_f} |1 + W_2^g(\omega)\cos(\tilde{k}x_0)|^2\,\mathrm{d}\omega, \tag{C1}$$

where

$$W_2^g(\omega) = \mathbf{1}_{[-\omega_g/2,\,\omega_g/2]}(\omega)\sum_{m=0}^{M_w} a_m\,\mathrm{e}^{(2m+1)\mathrm{j}kx_0}. \tag{C2}$$

The coefficients a_m are given by equation (50). Hence

$$J(W_2^g) = A_p\left(\frac{\rho_0 C_0}{S}\right)^2 \mathrm{e}^{-2\alpha x}\left\{\frac{2}{t_f} - \frac{2}{t_{\max}} + \frac{1}{2\pi}\int_{-\omega_{\min}}^{\omega_{\min}}\left|1 + \cos(\tilde{k}x_0)\sum_{m=0}^{M_w} a_m\,\mathrm{e}^{(2m+1)\mathrm{j}kx_0}\right|^2\mathrm{d}\omega\right\}, \tag{C3}$$

with $\omega_{\min} = \min(\omega_g/2, \omega_f)$ and $t_{\max} = \max(2t_g, t_f)$. Let $B(\omega)$ denote the expression $|1 + \cos(\tilde{k}x_0)\,\Sigma_{m=0}^{M_w} a_m\,\mathrm{e}^{(2m+1)\mathrm{j}kx_0}|^2$ to be integrated. To simplify the notations, one can introduce also the coefficient a_{M_w+1}, which is equal to zero. One then has

$$B(\omega) = \left|1 + \frac{1}{2}\sum_{m=0}^{M_w} a_m(\mathrm{e}^{-\alpha x_0}\,\mathrm{e}^{(2m+2)\mathrm{j}kx_0} + \mathrm{e}^{\alpha x_0}\,\mathrm{e}^{2m\mathrm{j}kx_0})\right|^2$$

$$= \left|1 + \frac{a_0}{2}\mathrm{e}^{\alpha x_0} + \frac{1}{2}\sum_{m=1}^{M_w+1}(a_m\,\mathrm{e}^{\alpha x_0} + a_{m-1}\,\mathrm{e}^{-\alpha x_0})\,\mathrm{e}^{2m\mathrm{j}kx_0}\right|^2$$

$$= \left(1 + \frac{a_0}{2}\mathrm{e}^{\alpha x_0}\right)^2 + \left(1 + \frac{a_0}{2}\mathrm{e}^{\alpha x_0}\right)\sum_{m=1}^{M_w+1}(a_m\,\mathrm{e}^{\alpha x_0} + a_{m-1}\,\mathrm{e}^{-\alpha x_0})\cos(2m\mathrm{j}kx_0)$$

$$+ \frac{1}{2}\sum_{m=1}^{M_w+1}\sum_{n=1}^{m+1}(a_m\,\mathrm{e}^{\alpha x_0} + a_{m-1}\,\mathrm{e}^{-\alpha x_0})(a_n\,\mathrm{e}^{\alpha x_0} + a_{n-1}\,\mathrm{e}^{-\alpha x_0})\cos(2(m-n)kx_0)$$

$$+ \frac{1}{4}\sum_{m=1}^{M_w+1}(a_m\,\mathrm{e}^{\alpha x_0} + a_{m-1}\,\mathrm{e}^{-\alpha x_0})^2. \tag{C4}$$

Suppose that the ratio $2t_0/t_{\max}$ is an integer so that the integral of equation (C3) can be simplified. If this ratio is not integer but large enough the terms that are cancelled are small with respect to the remaining terms.

$$\frac{1}{2\pi}\int_{-\omega_{\min}}^{\omega_{\min}} B(\omega)\,\mathrm{d}\omega = \frac{2}{t_{\max}}\left\{\left(1+\frac{a_0}{2}\,\mathrm{e}^{\alpha x_0}\right)^2 + \frac{1}{4}\sum_{m=1}^{M_w+1}(a_m\,\mathrm{e}^{\alpha x_0} + a_{m-1}\,\mathrm{e}^{-\alpha x_0})^2\right\}. \tag{C5}$$

One can develop the series of equation (C5) and use the relations (42) to simplify it:

$$\begin{aligned}
&\sum_{m=1}^{M_w+1}(a_m\,\mathrm{e}^{\alpha x_0} + a_{m-1}\,\mathrm{e}^{-\alpha x_0})^2\\
&\quad = 2\sum_{m=1}^{M_w+1} a_m a_{m-1} + \mathrm{e}^{2\alpha x_0}\sum_{m=1}^{M_w+1} a_m^2 + \mathrm{e}^{-2\alpha x_0}\sum_{m=1}^{M_w+1} a_{m-1}^2\\
&\quad = \sum_{m=1}^{M_w+1} a_m a_{m-1} + \sum_{m=0}^{M_w} a_m a_{m+1} + \mathrm{e}^{2\alpha x_0}\sum_{m=1}^{M_w+1} a_m^2 + \mathrm{e}^{-2\alpha x_0}\sum_{m=0}^{M_w} a_m^2\\
&\quad = a_0(\mathrm{e}^{-2\alpha x_0}a_0 + a_1) + \sum_{m=1}^{M_w} a_m(a_{m-1} + 2\,\mathrm{ch}\,(2\alpha x_0)a_m + a_{m+1})\\
&\quad = -2a_0\,\mathrm{e}^{\alpha x_0} - a_0^2\,\mathrm{e}^{2\alpha x_0}.
\end{aligned} \tag{C6}$$

Equation (C5) can then be simplified:

$$\frac{1}{2\pi}\int_{-\omega_{\min}}^{\omega_{\min}} B(\omega)\,\mathrm{d}\omega = \frac{2}{t_{\max}}\left\{\left(1+\frac{a_0}{2}\,\mathrm{e}^{\alpha x_0}\right)^2 - \frac{a_0}{2}\,\mathrm{e}^{\alpha x_0} - \frac{a_0^2}{4}\,\mathrm{e}^{2\alpha x_0}\right\} = \frac{2}{t_{\max}}\left\{1+\frac{a_0}{2}\,\mathrm{e}^{\alpha x_0}\right\}. \tag{C7}$$

One can now rewrite $J(W_2^g)$:

$$\begin{aligned}
J(W_2^g) &= A_p\left(\frac{\rho_0 C_0}{S}\right)^2 \mathrm{e}^{-2\alpha x}\left\{\frac{2}{t_f} - \frac{2}{t_{\max}} + \frac{2}{t_{\max}}\left(1+\frac{a_0}{2}\,\mathrm{e}^{\alpha x_0}\right)\right.\\
&= A_p\left(\frac{\rho_0 C_0}{S}\right)^2 \mathrm{e}^{-2\alpha x}\left\{\frac{2}{t_f} + \frac{a_0}{t_{\max}}\,\mathrm{e}^{\alpha x_0}\right\}\\
&= A_p\left(\frac{\rho_0 C_0}{S}\right)^2 \mathrm{e}^{-2\alpha x}\left\{\frac{2}{t_f} - \frac{2}{t_{\max}}\frac{\mathrm{sh}\,[2(M_w+1)\alpha x_0]}{\mathrm{sh}\,[2(M_w+2)\alpha x_0]}\,\mathrm{e}^{2\alpha x_0}\right\}.
\end{aligned} \tag{C8}$$

APPENDIX D: VARIATIONS OF γ WITH T_g

Suppose that V_{DSP} is greater than 12 and that the non-dimensional sampling period T_g is greater than 0·5. One can then show that the acoustic attenuation decreases with T_g.

Consider T_g in the segment $[\sqrt{3(2M_w+1)/V_{DSP}}, \sqrt{3(2M_w+3)/V_{DSP}}]$ where M_w is an integer. This case corresponds to a non-dimensional time length in the segment $[2M_w+1, 2M_w+3]$. Inside this segment the acoustic attenuation decreases with T_g since $G_w(\frac{1}{3}T_g^2 V_{DSP}, \alpha_0)$ is constant and $G_g(T_g)$ decreases with T_g.

Compare now the acoustic attenuation for the bounds of the segment. With the notation $G_{M_w} = G_g(\sqrt{3(2M_w + 1)/V_{DSP}})G_w(2M_w + 1, \alpha_0)$ and the ratio G_{M_w}/G_{M_w+1}, one has

$$\frac{G_{M_w}}{G_{M_w+1}} = \frac{\text{sh}\,[2(M_w + 1)\alpha_0]\,\text{sh}\,[2(M_w + 3)\alpha_0]}{\text{sh}^2\,[2(M_w + 2)\alpha_0]}\sqrt{\frac{2M_w + 3}{2M_w + 1}}\,. \tag{D1}$$

One can show that the ratio G_{M_w}/G_{M_w+1} is greater than 1, as follows.

With $g(\alpha_0) = \text{sh}\,[2(M_w + 1)\alpha_0]\,\text{sh}\,[2(M_w + 3)\alpha_0]/\text{sh}^2\,[2(M_w + 2)\alpha_0]$ one can show first that $g(\alpha_0) \geqslant g(0)$ for $\alpha_0 \geqslant 0$:

$$g(\alpha_0) = \frac{e^{4(M_w+2)\alpha_0} + e^{-4(M_w+2)\alpha_0} - e^{4\alpha_0} - e^{-4\alpha_0}}{e^{4(M_w+2)\alpha_0} + e^{-4(M_w+2)\alpha_0} - 2}$$

$$= 1 - \frac{1 - \text{ch}\,[4\alpha_0]}{1 - \text{ch}\,[4(M_w + 2)\alpha_0]} = 1 - \frac{\text{sh}^2\,[2\alpha_0]}{\text{sh}^2\,[2(M_w + 2)\alpha_0]}\,. \tag{D2}$$

With the notation $f(\alpha_0) = \text{sh}\,[2\alpha_0]/\text{sh}\,[2(M_w + 2)\alpha_0]$ and $h(\alpha_0) = \text{sh}\,[2\alpha_0] - \text{sh}\,[2(M_w + 2)\alpha_0]$, the derivative of $h(\alpha_0)$ is

$$(\text{d}h/\text{d}\alpha_0) = 2\text{ch}\,[2\alpha_0] - 2(M_w + 2)\,\text{ch}\,[2(M_w + 2)\alpha_0]. \tag{D3}$$

The derivative of $h(\alpha_0)$ is negative for all α_0. The function $h(\alpha_0)$ decreases therefore with α_0. Consider positive real numbers α_1 and α_2 such that $\alpha_2 \geqslant \alpha_1$. Then

$$\text{sh}\,[2\alpha_2] - \text{sh}\,[2(M_w + 2)\alpha_2] \leqslant \text{sh}\,[2\alpha_1] - \text{sh}\,[2(M_w + 2)\alpha_1], \tag{D4}$$

$$f(\alpha_2) - 1 \leqslant \frac{\text{sh}\,[2(M_w + 2)\alpha_1]}{\text{sh}\,[2(M_w + 2)\alpha_2]}\{f(\alpha_1) - 1\} \leqslant f(\alpha_1) - 1. \tag{D5}$$

Hence

$$f(\alpha_2) \leqslant f(\alpha_1). \tag{D6}$$

The function $f(\alpha_0)$ decreases and $g(\alpha_0)$ increases with α_0. One deduces that

$$g(\alpha_0) \geqslant g(0) \quad \forall\alpha_0 \geqslant 0. \tag{D7}$$

With equation (D5) one can now write

$$\frac{G_{M_w}}{G_{M_w+1}} \geqslant \frac{(M_w + 1)(M_w + 3)}{(M_w + 2)^2}\sqrt{\frac{2M_w + 3}{2M_w + 1}}\,. \tag{D8}$$

Since for all positive integers M_w,

$$\frac{(M_w + 1)(M_w + 3)}{(M_w + 2)^2}\sqrt{\frac{2M_w + 3}{2M_w + 1}} = \sqrt{\frac{2M_w^5 + 19M_w^4 + 68M_w^3 + 114M_w^2 + 90M_w + 27}{2M_w^5 + 17M_w^4 + 56M_w^3 + 88M_w^2 + 64M_w + 16}} > 1, \tag{D9}$$

one concludes that

$$G_{M_w}/G_{M_w+1} > 1. \tag{D10}$$

The acoustic attenuation decreases therefore with the non-dimensional sampling period T_g.